

CLONES ABOVE THE UNARY CLONE

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ABSTRACT. Let $\mathfrak{c} := 2^{\aleph_0}$. We give a family of pairwise incomparable clones on \mathbb{N} with $2^{\mathfrak{c}}$ members, all with the same unary fragment, namely the set of all unary operations.

We also give, for each n , a family of $2^{\mathfrak{c}}$ clones all with the same n -ary fragment, and all containing the set of all unary operations.

1. INTRODUCTION

In this paper, X will always be a countably infinite set. For a fixed base set X , an operation on X is a function $f : X^n \rightarrow X$ for some positive natural number n . A clone on X is a set of operations that contains all projection functions and is closed under composition. The set of all clones on X ordered by inclusion forms a complete lattice. (The survey paper [3] gives some background about clones, and in particular collects many recent concerning clones on infinite sets.)

We write $\mathcal{O}^{(n)}$ for the set X^{X^n} of all n -ary operations. For a clone C , call $C^{(n)} := C \cap \mathcal{O}^{(n)}$ the “ n -ary fragment of C ”.

$C^{(1)}$ is a submonoid of the monoid X^X of all unary operations. For any monoid $M \subseteq X^X$ the set of all clones C with $C^{(1)} = M$ is called the monoidal interval of M ; it has a least element, the clone generated by M , and a largest element $\text{Pol}(M)$, the set of all operations f satisfying $f(m_1, \dots, m_k) \in M$ whenever $m_1, \dots, m_k \in M$. (Here $f(m_1, \dots, m_k)$ is the unary operation mapping x to $f(m_1(x), \dots, m_k(x))$).

In [2], we showed that on $X = \mathbb{N}$ there are uncountably many clones containing all unary operations (but only two coatoms, see [1], [4]); in other words, the monoidal interval of X^X is uncountable. Pinsker in [6] has constructed (on arbitrary infinite base sets X) different monoids whose monoidal interval have various sizes, among them also one whose monoidal interval has size $2^{2^{|X|}}$.

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We will show here that (for $|X| = \aleph_0$) the interval associated with the monoid of all $f : X \rightarrow X$ has the largest possible size: 2^c . We will also construct, for any natural number $n \geq 1$, many clones which share their n -ary fragment with 2^c other clones.

Local clones. A clone C is local if it is closed in the natural topology on \mathcal{O} , that is: for all $f \in \mathcal{O}^{(k)} \setminus C$ there is a finite set $A \subseteq X^k$ such that no $g \in C$ agrees with f on A .

For any infinite set X , the local clones on X which contain all unary operations are precisely known: they form an increasing chain of order type $\omega + 1$.

1.1. Main results.

Theorem 1.1. *Let $X = \mathbb{N}$ be countably infinite. Then there are 2^c clones on X containing the monoid of all unary operations.*

To generalize the theorem also to larger arities, we need the following technical definition:

Definition 1.2. Let $\alpha \in \mathbb{R}$. A operation $f : X^d \rightarrow X$ is defined to be α -modest iff for all natural numbers N and all $Y \subseteq X$ of cardinality N , the range of $f|Y^d$ has at most αN elements.

f is modest if f is α -modest for some α .

We call a clone C modest if all operations in C are modest.

We write \mathcal{M} for the set of all modest operations.

Note that \mathcal{M} is a clone (the greatest modest clone), and that all unary operations are modest; in addition, all operations with finite range are modest, as well.

Theorem 1.3. *Let $d \geq 1$, and let C be a modest clone on \mathbb{N} containing all d -ary operations with range $\{0, 1\}$. Then there are 2^c many clones D with $D \cap \mathcal{O}^{(d)} = C \cap \mathcal{O}^{(d)}$.*

Taking $d = 1$ and C the clone of all essentially unary operations, we get theorem 1.1 as a special case.

Machida [5] has defined a natural metric on clones: The distance between two clones is $1/n$, where n is minimal with $C \cap \mathcal{O}^{(n)} \neq D \cap \mathcal{O}^{(n)}$. In this language, theorem 1.3 says that certain sets of clones can be arbitrarily small from the metric/topological point of view, and still large when measured by cardinality.

Let F be a set of operations. We write $\langle F \rangle$ for the smallest clone containing F . If C is a clone, we may write $\langle F \rangle_C$ instead of $\langle F \cup C \rangle$ (and for $F = \{f, g, \dots\}$ we write $\langle f, g, \dots \rangle_C$ instead of $\langle \{f, g, \dots\} \rangle_C$. Note that $f \in \langle F \rangle_C$ iff there is a finite subset $F_0 \subseteq F$ with $f \in \langle F_0 \rangle_C$.

Both sections of this paper use the following easy fact:

Lemma 1.4. *Let C be a clone, and let $(f_i : i \in I)$ be a family of operations which is independent over C (which means that $f_i \notin \langle f_j : j \neq i \rangle_C$ for all $i \in I$). For $J \subseteq I$ let $C_J = \langle f_i : i \in J \rangle_C$. Then:*

- (a) *The map $J \mapsto C_J$ is a 1-1 order-preserving map from $\mathfrak{P}(I)$, the power set of I , into the interval $[C, \langle f_i : i \in I \rangle]$ in the clone lattice (both ordered by inclusion).*
- (b) *If I has cardinality κ , then $\{C_J : J \subseteq I\}$ contains 2^κ many elements, and it is order-isomorphic with $\mathfrak{P}(I)$.*
- (c) *Assume moreover that $\{f_i : i \in I\} \subseteq \text{Pol}(C \cap \mathcal{O}^{(d)})$. Then $C_J \cap \mathcal{O}^{(d)} = C \cap \mathcal{O}^{(d)}$ for all $J \subseteq I$.*

Proof. (a) and (b) are clear.

For (c), recall that $\text{Pol}(C \cap \mathcal{O}^{(d)})$ is the set of all operations f with $f(c_1, \dots, c_m) \in C \cap \mathcal{O}^{(d)}$ whenever $c_1, \dots, c_m \in C \cap \mathcal{O}^{(d)}$. Clearly the assumption implies

$$C \subseteq \langle f_i : i \in I \rangle_C \subseteq \text{Pol}(C \cap \mathcal{O}^{(d)}),$$

and by definition, the clones C and $\text{Pol}(C \cap \mathcal{O}^{(d)})$ have the same d -ary fragment D . Consequently, the d -ary fragment of C_J is D , as well. \square

2. SPARSE GRAPHS AND MODEST OPERATIONS

Definition 2.1. Let (V, E) be a graph (i.e., $E \subseteq [V]^2$, where $[V]^2$ is the set of 2-element subsets of V).

We say that (V, E) is (k, l) -sparse, if for every $U \subseteq V$ of size at most k , the induced subgraph on U has at most l edges.

In order to help the reader, in this paragraph we are providing a brief and informal explanation for the technical details of the rest of this section. In Lemma 2.3 below we will show, that for large enough N and $0 < \varepsilon < \frac{1}{2}$, there exist graphs \mathcal{G} on N vertices, whose M -sized subgraphs (for small M) are sparse, while at the same time, these \mathcal{G} have “many” edges: the number of their edges is at least $N^{1+\varepsilon}$. Using this lemma, we will be able to construct functions on finite domains, which have large range, but the range of their restrictions to small sets remain small; for the details see Lemma 2.6. “Gluing together” carefully an infinite sequence of such operations, we obtain a set S of operations on \mathbb{N} such that S is independent and has cardinality \mathfrak{c} . Combining this with Lemma 1.4, the proof of Theorem 1.1 will follow quickly.

Definition 2.2. Let M, N be natural numbers, $0 < \varepsilon < \frac{1}{2}$. We write $M \ll_\varepsilon N$ if $M \cdot N^{2\varepsilon-1} < 1/10$.

Lemma 2.3. *Let $\varepsilon < 1/2$, and let $1 \leq M \ll_\varepsilon N$. Then there is a graph $G = (V, E)$ with N vertices and more than $N^{1+\varepsilon}$ edges which is $(k, 2k)$ -sparse for all $k \leq M$.*

Proof. We will use an Erdős type probability argument: we will define a suitable probability measure on all graphs on N vertices and then show that the set of graphs not satisfying the conclusion has small measure.

We note, that a somewhat stronger form of the lemma follows quickly from the Central Limit Theorem. For completeness, we present an elementary proof.

Let $p := 4N^{-1+\varepsilon}$, and let μ be the probability measure on $\{0, 1\}$ with $\mu(\{1\}) = p$. Fix a set V of N vertices; there are $\frac{N(N-1)}{2}$ potential edges. Via characteristic functions, we identify the set of all graphs on V with the product space $\{0, 1\}^{\frac{N(N-1)}{2}}$, which is equipped with the product probability structure. In order to keep notation simpler, the product measure will also be called μ .

In other words, for each potential edge e we flip a weighted coin (independent of all other coin flips), and with probability p will decide to add e to our graph. The expected number of edges is $\frac{N(N-1)}{2} \cdot p \approx 2N^{1+\varepsilon}$, with variance $\frac{N(N-1)}{2} p(1-p) \approx 2N^{1+\varepsilon}$. By Chebyshev's inequality, most graphs will have more than $N^{1+\varepsilon}$ edges. More precisely, the measure of the set of graphs with fewer than $N^{1+\varepsilon}$ edges is smaller than

$$\frac{\frac{N(N-1)}{2} p(1-p)}{(\frac{N(N-1)}{2} \cdot p - N^{1+\varepsilon})^2} \approx \frac{2N^{1+\varepsilon}}{(N^{1+\varepsilon})^2} = 2N^{-1-\varepsilon} < 1/2,$$

because, by the assumptions of the lemma, we have $4 \leq N$.

We now estimate the measure of the set \mathcal{G} of all graphs on V which are not $(k, 2k)$ -sparse for some $k \leq M$.

For any set $E' \subseteq [V]^2$ we let $\mathcal{G}_{E'}$ be the set of all graphs whose edges include the set E' . Clearly $\mu(\mathcal{G}_{E'}) = (4N^{-1+\varepsilon})^{|E'|}$.

For each graph (V, E) which is not $(k, 2k)$ -sparse there exists a set V' of k vertices, and a set $E' \subseteq [V']^2$ with $2k$ elements such that $E \supseteq E'$, i.e., $(V, E) \in \mathcal{G}_{E'}$. So the measure of all those graphs is bounded above by

$$\sum_{\substack{V' \subseteq V \\ |V'|=k}} \sum_{\substack{E' \subseteq [V']^2 \\ |E'|=2k}} \mu(\mathcal{G}_{E'}).$$

The crucial component in this sum is the summation over all subsets of size k ; this will be estimated by a factor N^k ; the other summations will be replaced by factors that depend on k only. Altogether we get an upper bound

$$N^k (k^2)^{2k} (4N^{-1+\varepsilon})^{2k} = (2k)^{4k} N^k N^{-2k(1-\varepsilon)} = (2k)^{4k} N^{k(2\varepsilon-1)} \approx N^{k(2\varepsilon-1)}.$$

Now summing over all $k \leq M$ yields

$$\sum_{k=1}^M N^{k(2\varepsilon-1)} \leq M \cdot N^{2\varepsilon-1} < 1/10$$

as $M \ll_{\varepsilon} N$.

Hence the set of graphs satisfying the conclusion has measure > 0 , so it is nonempty. \square

Lemma 2.4. *Let $0 < \varepsilon < \frac{1}{2}$. There is an increasing sequence $\langle N_\ell : \ell \in \mathbb{N} \rangle$ of natural numbers and a sequence $\langle (V_\ell, E_\ell) : \ell \in \mathbb{N} \rangle$ of graphs such that:*

- (1) $\max\{N_\ell^2 + 1, 2^{3N_\ell}, 1 + |E_\ell|\} < N_{\ell+1}$.
- (2) $V_\ell = [N_{\ell-1}, N_\ell]$.
- (3) $|E_\ell| \geq N_\ell^{1+\varepsilon}$.
- (4) For all $k \leq 2^{\ell+1}N_{\ell-1}$, the graph (V_ℓ, E_ℓ) is $(k, 2k)$ -sparse.

Proof. We can choose N_ℓ by recursion; given $N_{\ell-1}$, Lemma 2.3 tells us how large N_ℓ has to be. In more detail, let $\varepsilon' > \varepsilon$ be such that $\varepsilon < \varepsilon' < \frac{1}{2}$. Then, by Lemma 2.3, there exist N'_ℓ and a graph \mathcal{G} with N'_ℓ vertices and more than $(N'_\ell)^{1+\varepsilon'}$ edges which is $(k, 2k)$ -sparse for all $k \leq 2^{\ell+1}N_{\ell-1}$. Enlarging N'_ℓ if necessary, we may assume, that

- (1) holds (more precisely, N'_ℓ is larger than the left hand side of the $(\ell - 1)^{th}$ instance of (1)), and
- $(1 + \varepsilon) \ln(2) < (\varepsilon' - \varepsilon) \ln(N'_\ell)$ and $2N_{\ell-1} \leq N'_\ell$.

Take $N_\ell := N_{\ell-1} + N'_\ell$. Let \mathcal{G}_ℓ be an isomorphic copy of \mathcal{G} with $V_\ell = [N_{\ell-1}, N_\ell]$. Now (2) and (4) of the statement clearly hold for \mathcal{G}_ℓ . To check (3), it is enough to show that $N_\ell^{1+\varepsilon} \leq (N'_\ell)^{1+\varepsilon'}$, that is,

$$(*) \quad \ln(N_\ell^{1+\varepsilon}) \leq \ln((N'_\ell)^{1+\varepsilon'}).$$

The following calculation proves (*):

$$\begin{aligned} \ln(N_\ell^{1+\varepsilon}) &= (1 + \varepsilon) \ln(N_{\ell-1} + N'_\ell) \\ &\leq (1 + \varepsilon) \ln(2N'_\ell) \\ &= (1 + \varepsilon) \ln(N'_\ell) + (1 + \varepsilon) \ln(2) \\ &\leq (1 + \varepsilon) \ln(N'_\ell) + (\varepsilon' - \varepsilon) \ln(N'_\ell) \\ &= (1 + \varepsilon') \ln(N'_\ell) \\ &= \ln((N'_\ell)^{1+\varepsilon'}) \end{aligned} \quad \square$$

So our graphs (V_ℓ, E_ℓ) have “many edges” on a large scale (i.e., looking at the whole graph), but only “few edges” on the small scale (looking at small induced subgraphs).

Definition 2.5. A ϱ -ary (partial) function $f : V^\varrho \rightarrow \mathbb{N}$ is defined to be (k, l) -modest iff for any $U_0, \dots, U_{\varrho-1} \subseteq V$ of size at most k , $f|_{(U_0 \times \dots \times U_{\varrho-1})}$ has at most l values.

The “support” of a function f is the set of elements in the domain of f where f is defined and its value is not equal to 0.

Lemma 2.6. *Let (V, E) be a graph which is $(k, 2k)$ -sparse for all $k \leq M$. Then there is a function $f : V \times V \rightarrow \mathbb{N}$ which has at least $|E|$ values but is $(k, 5k)$ -modest for all $k \leq M/2$.*

Proof. Let f be a symmetric function which takes different values on all edges in E , and is constantly zero outside E . Then for each $U_1, U_2 \subseteq V$ of size $k \leq M/2$, $E \upharpoonright (U_1 \cup U_2)^2$ has at most $2 \cdot 2k$ edges, so f can take at most $4k + 1$ values on $U_1 \times U_2 \subseteq (U_1 \cup U_2)^2$. \square

Corollary 2.7. *There is an increasing sequence $\langle N_\ell : \ell \in \mathbb{N} \rangle$ of natural numbers and a sequence $\langle s_\ell : \ell \in \mathbb{N} \rangle$ of operations $s_\ell : [N_{\ell-1}, N_\ell]^2 \rightarrow \mathbb{N}$ satisfying the following:*

- (1) $N_\ell^2 + 1 < N_{\ell+1}$ and also $2^{N_\ell} < \sqrt[3]{N_{\ell+1}}$.
- (2) Each s_ℓ is $(k, 5k)$ -modest for all $k \leq 2^\ell N_{\ell-1}$.
- (3) Each s_ℓ is $(k, 5k)$ -modest for $k \geq N_{\ell+1}$.
- (4) For all ℓ , the range of s_ℓ has more than $N_\ell^{4/3}$ elements.

Proof. Let $\varepsilon = \frac{1}{3}$ and let $\langle N_\ell : \ell \in \mathbb{N} \rangle$ and $\langle (V_\ell, E_\ell) : \ell \in \mathbb{N} \rangle$ be the sequences obtained from Lemma 2.4. In addition, for every $\ell \in \mathbb{N}$, let s_ℓ be the operation obtained from (V_ℓ, E_ℓ) by Lemma 2.6. We claim that this choice satisfies the statement.

(1) follows from Lemma 2.4(1). Combining Lemma 2.4(4) with Lemma 2.6 one obtains (2). By the construction described in Lemma 2.6, the range of s_ℓ has cardinality at most $|E_\ell| + 1 < N_{\ell+1}$. Hence (3) holds trivially because of Lemma 2.4(1). Finally, (4) follows from Lemma 2.4(3) (combined with the choice of ε and with Lemma 2.6). \square

From now on we fix sequences $\langle N_\ell : \ell \in \mathbb{N} \rangle$ and $\langle s_\ell : \ell \in \mathbb{N} \rangle$ as above.

Definition 2.8. For every $A \subseteq \mathbb{N}$ let $s_A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be defined from s_ℓ as follows: s_A is $\bigcup_{\ell \in A} s_\ell$, extended by the value 0 wherever it is undefined (i.e., $s_A \upharpoonright [N_{\ell-1}, N_\ell] \times [N_{i-1}, N_i]$ is constantly zero for $\ell \neq i$).

Lemma 2.9.

- (1) If $\ell < i$, then s_i is $(k, 5k)$ -modest for all $k \leq 2^\ell N_\ell$.
- (2) If $\ell \notin A$, then s_A is $(k, 6k)$ -modest for all k in $[N_\ell, 2^\ell N_\ell]$.

Proof. First we prove (1). By Lemma 2.7(2), s_i is $(k, 5k)$ -modest for all $k \leq 2^i N_{i-1}$, so certainly also for all $k \leq 2^\ell N_\ell$.

Now we prove (2). Let X, Y be sets of size k , with k in $[N_\ell, 2^\ell N_\ell]$. Let $X_- = X \cap N_\ell$, $X_+ = X \setminus X_-$, and define Y_-, Y_+ similarly. We have

$$s_A[X \times Y] \subseteq s_A[X_- \times Y_-] \cup s_A[X_+ \times Y_+] \cup \{0\}.$$

- Because $\ell \notin A$, s_A is constantly 0 on $(X_- \times Y_-) \setminus (N_{\ell-1} \times N_{\ell-1})$. Hence the first set has size at most $N_{\ell-1}^2 \leq N_\ell - 1 \leq k - 1$.
- To estimate the size of $s_A[X_+ \times Y_+] \cup \{0\} \subseteq \{0\} \cup \bigcup_{i > \ell} s_i[X_+ \times Y_+]$ we use (1) to see that this set is bounded by $5k + 1$.
- So $s_A[X \times Y]$ has size at most $6k$.

\square

Definition 2.10. Let $A_1, \dots, A_n \subseteq \mathbb{N}$. A (binary) *term* in the operations s_{A_1}, \dots, s_{A_n} is a formal expression involving (some of) the variables x, y , (some of) the operations s_{A_1}, \dots, s_{A_n} , as well as any unary operations. (We trust the reader to supply a formal definition by induction.)

The *depth* of a term τ is defined inductively as follows:

- x and y have depth 0.
- For any unary operation u , the depth of $u(\tau)$ is 1 more than the depth of τ .
- Let m be the maximum of the depths of τ_1 and τ_2 . Then the depth of $f_{A_i}(\tau_1, \tau_2)$ is $m + 1$.

Every term naturally induces a binary operation on \mathbb{N} . (Note that the same operation may be represented by different terms, even terms of different depths.)

Lemma 2.11. *Let τ be a term in the operations s_{A_1}, \dots, s_{A_n} of depth d . Let $\ell > d \log_2(6)$, and assume $\ell \notin A_1 \cup \dots \cup A_n$. Then we have:*

- (1) *The operation represented by τ is $(N_\ell, 6^d N_\ell)$ -modest.*
- (2) *In particular, τ cannot represent the operation s_ℓ , or s_B for any B containing ℓ .*

Proof. We start to show (1) by induction on d (or more precisely, on τ).

- If τ is x or y , this is trivial.
- If $\tau = u(\tau_1)$ then again the range of $u(\tau_1)$ is not larger than the range of τ_1 .
- Assume $\tau = s_{A_i}(\tau_1, \tau_2)$, where the depths of τ_1 and τ_2 are at most d . Observe that
 - Both τ_1 and τ_2 are $(N_\ell, 6^d N_\ell)$ -modest by the inductive assumption.
 - By Lemma 2.9(2), f_{A_i} is $(6^d N_\ell, 6 \cdot 6^d N_\ell)$ -modest. (Recall that $d \log_2(6) \leq \ell$, so $6^d N_\ell \leq 2^\ell N_\ell$.)

Now let $U_1, U_2 \subseteq \mathbb{N}$ be two sets, both of size at most N_ℓ . Then, according to the previous observation, the ranges of $\tau_1|U_1$ and $\tau_2|U_2$ have size at most $6^d N_\ell$. Hence, again by the previous observation, the cardinality of the range of $\tau|U_1 \times U_2$ is at most $6 \cdot 6^d N_\ell = 6^{d+1} N_\ell$, as desired.

Now we turn to prove (2). By assumption, $6^d \leq 2^\ell \leq 2^{N_{\ell-1}}$, so by (1) of Corollary 2.7 we have $6^d N_\ell < N_\ell^{4/3}$. Hence, by (1) of the present theorem, $|\text{ran}(\tau|N_\ell \times N_\ell)| \leq 6^d N_\ell < N_\ell^{4/3}$, while, according to Corollary 2.7 (4), we have $|\text{ran}(s_B|N_\ell)| > N_\ell^{4/3}$.

□

Corollary 2.12. *Let B, A_1, \dots, A_n be pairwise distinct subsets of \mathbb{N} such that $B \setminus (A_1 \cup \dots \cup A_n)$ is infinite. Then $s_B \notin \langle s_{A_1}, \dots, s_{A_n} \rangle_{\mathcal{C}(1)}$.*

Proof. Assume, seeking a contradiction, that $s_B \in \langle s_{A_1}, \dots, s_{A_n} \rangle_{\mathcal{O}(1)}$. Then there exists a term τ in A_1, \dots, A_n representing s_B . Let d be the depth of τ . Then there exists $\ell \in B \setminus (A_1 \cup \dots \cup A_n)$ with $\ell > d \log_2(6)$. Then, by Lemma 2.11(2), τ does not represent s_B . This contradiction completes the proof. \square

Fact 2.13. There exists an independent family $(A_r : r \in \mathbb{R})$ of \mathfrak{c} subsets of \mathbb{N} . That is, for all disjoint finite subsets $J_+, J_- \subseteq \mathbb{R}$ the set $\bigcap_{r \in J_+} A_r \cap \bigcap_{r \in J_-} (\mathbb{N} \setminus A_r)$ is nonempty and even infinite.

Proof. This is well known. For example, replacing the base set \mathbb{N} by $\mathbb{Q}[x]$, the set of all polynomials with rational coefficients, we can take $A_r := \{p(x) \in \mathbb{Q}[x] : p(r) > 0\}$. \square

Proof of theorem 1.1. Choose an independent family $(A_r : r \in \mathbb{R})$ of subsets of \mathbb{N} . Then, for all finite $S \subseteq \mathbb{R}$ and all $r \in \mathbb{R} \setminus S$ the set $A_r \setminus \bigcup_{s \in S} A_s$ is infinite. By Corollary 2.12, $\{s_{A_r} : r \in \mathbb{R}\}$ is a family of operations independent over $\mathcal{O}^{(1)}$: for any $r \in \mathbb{R}$, we have $s_{A_r} \notin \langle s_{A_p} : p \in \mathbb{R} \setminus \{r\} \rangle_{M \cap \mathcal{O}^{(d)}}$. By Lemma 1.4 we are done. \square

3. HIGHER ARITIES

Definition 3.1. We say that an operation $f : X^d \rightarrow X$ is *modest* if there is some k such that for all $N > 1$, f is (N, kN) -modest, i.e.:

$f[X_1 \times \dots \times X_d]$ has at most kN elements
whenever each set $X_i \subseteq X$ has at most N elements.

We call a clone C modest if all operations in C are modest.

Note that the set of all modest operations is a clone (the greatest modest clone), and that all unary operations are modest, as are all operations with finite range.

The main result of the present section is as follows.

Theorem 3.2. *Let $d \geq 1$, and let C be a modest clone containing all d -ary operations with range $\{0, 1\}$. Then there are $2^{\mathfrak{c}}$ many clones D with $D \cap \mathcal{O}^{(d)} = C \cap \mathcal{O}^{(d)}$.*

We postpone the proof of this theorem to the end of this section. The number d will be fixed through this section.

Definition 3.3.

- We fix a language with object variables x_i , $i \in \mathbb{N}$ and formal operation variables \mathbf{f}_j^i ($i, j \in \mathbb{N}$), where the superscript i denotes the formal arity of \mathbf{f}_j^i . Terms are defined as usual: each object variable is a term, and whenever t_1, \dots, t_i are terms and $j \in \mathbb{N}$, then $\mathbf{f}_j^i(t_1, \dots, t_i)$ is a term, as well.

- The set of all terms can be enumerated as $\{\tau_1, \tau_2, \dots\}$ such that τ_m contains at most m occurrences of operation symbols, and each operation symbol occurring in τ_m is at most m -ary.
- Let τ be a term. We say that a sequence of functions $\bar{g} = (g_j^i : (i, j) \in S)$ is suitable for τ iff each g_j^i has arity i , and $(i, j) \in S$ whenever the variable \mathbf{f}_j^i appears in τ .
- If \bar{g} is suitable for τ , then plugging in the g_j^i for the \mathbf{f}_j^i will yield a d -ary operation on X which we denote by $\tau[\bar{g}]$.

Definition 3.4. Let $d \geq 2$. For any set V we let $[V]^d$ be the set of d -element subsets of V . The structure (V, E) is defined to be a d -uniform hypergraph iff $E \subseteq [V]^d$. The elements of E are called the *hyperedges* of (V, E) .

Every $V' \subseteq V$ naturally induces a hypergraph $(V', E \cap [V']^d)$, which we may also denote by $(V', E|V')$.

We say that (V, E) is (k, l) -sparse, iff for every $X \subseteq V$ of size at most k , the hypergraph $(X, E|X)$ has at most l hyperedges.

Lemma 3.5. Fix d, k, ε . Let V be a set of cardinality N , let (V, E) be a $(d+1)$ -uniform hypergraph with at least $N^{d+\varepsilon}$ hyperedges. If N is large enough, then there is an operation $s : V^{d+1} \rightarrow V$ whose support is contained in E and whose values are in $\{0, 1\}$ such that for any set W with $V \subseteq W, |W| \leq kN$ the following holds.

Whenever $\tau \in \{\tau_1, \dots, \tau_k\}$,
 $\bar{g} = (g_j^i)_{i,j}$ is a suitable sequence of operations for τ on W
 with each g_j^i being

- either of arity at most d
- or of arity $d+1$ with support of size at most $3N \log N$

then $\tau[\bar{g}]$ does not represent s . In particular, there exists $e \in E$ such that s and $\tau[\bar{g}]$ have different values on e .

If N satisfies the above conditions, then we will say that N is k -large.

Proof. Let W be a set containing V with $|W| = kN$. Clearly, it is enough to show that there exists an operation $s : V^{d+1} \rightarrow V$ satisfying the statement for this particular W . There are only $(kN)^{(kN)^d}$ d -ary operations on W , and only k terms to be considered. A support is a subset of $[W]^{d+1}$; there are fewer than $\binom{(kN)^{d+1}}{3N \log N} \leq (kN)^{3N \log(N)(d+1)}$ possible supports of size $3N \log N$. For each support of size $3N \log N$ there are at most $(kN)^{3N \log N}$ possible operations that have this support. By the enumeration fixed in Definition 3.3, each term τ_i ($i \leq k$) contains at most k many operation variables. Counting the possibilities of choosing k many d -ary operations and k many $d+1$ -ary operations with support of size at most $3N \log N$, one can see, that altogether there are fewer than

$$t := (kN)^{(kN)^d \cdot k} \cdot k \cdot (kN)^{3N \log(N)(d+1)k} \cdot (kN)^{3N \log(N)k}$$

operations represented by such terms. We may assume $k \leq \log N$. Estimating k by N or by $\log N$, one obtains

$$t \leq (\log N) \cdot (N \cdot \log N)^{\log N \cdot (N \log N)^d} \cdot N^{6N(d+1) \log^2 N} \cdot N^{6N \log^2 N}.$$

Recall, that for any $\delta > 0$, $d \in \mathbb{N}$ and for large enough N , one has $\log^d N \leq N^\delta$. Let $0 < \delta < \varepsilon$. Then for large enough N , each of the four factors of t can be estimated by $N^{\frac{1}{4} \cdot N^{d+\delta}}$. Consequently, for large enough N , we have $t < N^{N^{d+\delta}} = 2^{N^{d+\delta} \cdot \log N}$. This number (for large enough N), is certainly less than $2^{N^{d+\varepsilon}}$.

But there are at least $2^{N^{d+\varepsilon}}$ possible operations on E with values in $\{0, 1\}$. So not all of them are representable. \square

Lemma 3.6. *Let $0 < \varepsilon < 1/2$. Then there are sequences $\bar{N} = \langle N_\ell : \ell < \mathbb{N} \rangle$, $\bar{E} = \langle E_\ell : \ell < \mathbb{N} \rangle$ with the following properties:*

- (1) \bar{N} is strictly increasing and in fact $N_{\ell-1}^{d+1} < N_\ell$, $2^\ell \leq N_\ell$ and N_ℓ is ℓ -large for all ℓ . We will write V_ℓ for the interval $[N_{\ell-1}, N_\ell)$.
- (2) (V_ℓ, E_ℓ) is a $(d+1)$ -uniform hypergraph with more than $N_\ell^{d+\varepsilon}$ hyperedges.
- (3) For every $k \leq N_{\ell-1}^2$, (V_ℓ, E_ℓ) is $(k, 2k)$ -sparse.

Proof. This proof is only a slight variation of the proof of Lemma 2.7, so we will be brief.

Assume $N_{\ell-1}$ has already been defined. We will choose N_ℓ after a certain amount of extra work such that $N_\ell \gg N_{\ell-1}$. Assume for a moment, that N_ℓ is already defined. Let $V_\ell := [N_{\ell-1}, N_\ell)$. Let J be the cardinality of the set $[V_\ell]^{d+1}$ of all potential hyperedges: $J = \binom{N_\ell - N_{\ell-1}}{d+1}$.

On the set of all $(d+1)$ -uniform hypergraphs (which we may identify with 2^J), we define a product measure by declaring the probability of each potential hyperedge to be $p := 2(d+1)! \cdot N_\ell^{\varepsilon-1}$.

So the expected number of hyperedges of a random hypergraph is $pJ = 2(d+1)! \cdot N_\ell^{\varepsilon-1} \cdot \binom{N_\ell - N_{\ell-1}}{d+1} \approx 2N_\ell^{\varepsilon-1} \cdot N_\ell^{d+1} = 2N_\ell^{d+\varepsilon}$. Again using Chebyshev's inequality, we see that with high probability a random hypergraph will have more than $N_\ell^{d+\varepsilon}$ hyperedges.

Now we estimate the probability that there is a sub-hypergraph with $k \leq N_{\ell-1}^2$ vertices which has more than $2k$ hyperedges, and we will show that it is very low.

For each potential k there are at most $\binom{N_\ell}{k} \leq N_\ell^k$ subsets; for each such subset S , the probability that a given set H of hyperedges with $j := |H| \geq 2k$ appears as a subset of $E \setminus S$ is $\leq p^j \leq p^{2k}$. There are $\binom{k^d}{j} \leq 2^{k^d}$ possibilities for H . So the probability that such a bad subgraph of size k exists is bounded from above by

$$N_\ell^k \cdot p^{2k} \cdot 2^{k^d}.$$

There are $N_{\ell-1}^2$ possibilities for k , so we have to choose N_ℓ such that

$$(*) \quad \sum_{k=1}^{N_{\ell-1}^2} N_\ell^k \cdot p^{2k} 2^{k^d} \leq \frac{1}{2}.$$

But $N_\ell^k \cdot p^{2k} \approx N_\ell^k N_\ell^{(\varepsilon-1)2k} = N_\ell^{k(2\varepsilon-1)}$ which converges to 0 if N_ℓ converges to infinity. Hence, one may choose N_ℓ so large, that

$$N_\ell^k \cdot p^{2k} < \frac{1}{N_{\ell-1}^2 \cdot 2^{(N_{\ell-1}^2)^d}}$$

and $N_\ell > \max\{2^\ell, N_{\ell-1}^{d+1}\}$ hold. Further increasing N_ℓ if necessary, we may choose it to be ℓ -large, as well. Estimating 2^{k^d} by $2^{(N_{\ell-1}^2)^d}$ in the left hand side of $(*)$, it follows, that the inequality in $(*)$ holds.

So the set of hypergraphs on V_ℓ which are not $(k, 2k)$ -sparse for some $k \leq N_{\ell-1}^2$ has measure at most $\frac{1}{2}$, while, almost all hypergraphs on V_ℓ have $N_\ell^{d+\varepsilon}$ hyperedges. It follows, that there exist N_ℓ and E_ℓ satisfying the requirements of the lemma, and thus, the sequences in the statement can be constructed recursively. \square

Definition 3.7. Let \bar{N} and \bar{E} be as in Lemma 3.6. For each $V_\ell = [N_{\ell-1}, N_\ell]$ let s_ℓ be a $(d+1)$ -ary operation with support E_ℓ which differs on E_ℓ from each $\tau_i[g]$ ($i \leq \ell$, \bar{g} as in Lemma 3.5).

For each infinite $A \subseteq \mathbb{N}$ let $s_A := \bigcup_{\ell \in A} s_\ell$ (where we replace all undefined values of s_A with 0).

Lemma 3.8. Let $B \subseteq \mathbb{N}$ be infinite, and assume $\ell \in \mathbb{N} \setminus B$. Let $W \subseteq \mathbb{N}$ be such that $|W| \leq \ell \cdot N_\ell$. Then the cardinality of the support of $s_B \upharpoonright W^{d+1}$ is at most $N_\ell(1 + 2 \log N_\ell)$.

Proof. Throughout this proof, the support of a function f is denoted by $\text{supp}(f)$. Let $W_1 = W \cap [0, N_{\ell-1}]$, $W_2 = W \cap [N_{\ell-1}, N_\ell]$, $W_3 = W \setminus (W_1 \cup W_2)$. By construction,

$$\begin{aligned} \text{supp}(s_B \upharpoonright W^{d+1}) &\subseteq \text{supp}(s_B \upharpoonright W_1^{d+1}) \cup \text{supp}(s_B \upharpoonright W_2^{d+1}) \cup \text{supp}(s_B \upharpoonright W_3^{d+1}) \\ &\bullet \text{ Clearly, } |\text{supp}(s_B \upharpoonright W_1^{d+1})| \leq N_{\ell-1}^{d+1} \text{ and } N_{\ell-1}^{d+1} \leq N_\ell \text{ by Lemma 3.6(1).} \\ &\bullet \text{ In addition, } \text{supp}(s_B \upharpoonright W_2^{d+1}) \text{ is empty because } \ell \notin B. \\ &\bullet \text{ Clearly, } |W_3| \leq |W| \leq \ell \cdot N_\ell \leq \log(N_\ell) N_\ell \text{ (in the last estimation we} \\ &\quad \text{used Lemma 3.6 (1): } \ell \leq \log N_\ell \text{). In addition, by Lemma 3.6 (3),} \\ &\quad \text{for any } j > \ell, (V_j, E_j) \text{ is } (N_\ell \log N_\ell, 2N_\ell \log N_\ell)\text{-sparse. It follows,} \\ &\quad \text{that } |\text{supp}(s_B \upharpoonright W_3^{d+1})| \leq 2N_\ell \log N_\ell. \end{aligned}$$

Combining these observations, the statement follows. \square

Lemma 3.9. If f_1, \dots, f_m are (k, k') -modest d -ary operations, g is a (k', k'') -modest m -ary operation, then $g(f_1, \dots, f_m)$ is (k, k'') -modest.

Proof. Easy. \square

Lemma 3.10. *Let \mathcal{M} be the clone of all modest operations. Let $A \setminus (B_1 \cup \dots \cup B_r)$ be infinite. Then $s_A \notin \langle s_{B_1}, \dots, s_{B_r} \rangle_{M \cap \mathcal{O}^{(d)}}$.*

Proof. For any term τ and any suitable sequence \bar{g} (consisting only of operations in $\langle (\mathcal{M} \cap \mathcal{O}^{(d)}) \cup \{s_{B_1}, \dots, s_{B_r}\} \rangle$) we will find $\ell \in A$ such that $\tau[g]$ disagrees with s_ℓ (hence also with s_A) on E_ℓ .

So fix a term $\tau = \tau_i$ and \bar{g} . Let ν be the number of subterms of τ and let k witness that all operations in \bar{g} are modest. Let $\ell > \nu \cdot k^i$ be in $A \setminus (B_1 \cup \dots \cup B_r)$. We claim that, for each subterm σ of τ (of depth s), the range of $\sigma[\bar{g}]$ over the domain V_ℓ^{d+1} has cardinality at most $N_\ell \cdot k^s$.

This can be proved by induction on the depth of σ , using Lemma 3.9 combined with the fact that the operations s_{B_j} take only 2 values, and that all other operations in \bar{g} are modest, witnessed by k .

Recall, that according to the enumeration fixed in Definition 3.3, the depth of $\tau = \tau_i$ is at most i . So the set of all intermediate values in the computation of $\tau[g]$ on E_ℓ has size at most $\nu \cdot k^i N_\ell < \ell N_\ell$. Let $W \supseteq V_\ell$ be a set of size at most ℓN_ℓ containing $\{0, 1\}$ and all these intermediate values. The term τ induces a partial function $\tau[\bar{g}] \upharpoonright E_\ell$. By replacing all values of the operations in \bar{g} by 0 if they are outside W , we get a sequence \bar{g}' of operations with the following properties:

- $\tau[\bar{g}']$ is a total function from W^{d+1} to W .
- $\tau[\bar{g}']$ agrees with $\tau[\bar{g}]$ on E_ℓ .
- All operations in \bar{g}' are either some s_{B_j} or an operation of arity at most d .

By Lemma 3.8, the support of each $s_{B_j} \upharpoonright W^{d+1}$ is at most $N_\ell(1 + 2 \log N_\ell) \leq 3N_\ell \log N_\ell$. So by the construction of s_ℓ , and by Lemma 3.5, s_ℓ disagrees with $\tau[\bar{g}']$ somewhere on E_ℓ ; so s_ℓ also disagrees with $\tau[\bar{g}]$. \square

Now we are ready to prove Theorem 3.2.

Proof of theorem 3.2. Similarly to the proof of Theorem 1.1, choose an independent family $(A_r : r \in \mathbb{R})$ of subsets of \mathbb{N} . Then, for all finite $S \subseteq \mathbb{R}$ and all $r \in \mathbb{R} \setminus S$ the set $A_r \setminus \bigcup_{s \in S} A_s$ is infinite. By Lemma 3.10, $\{s_{A_r} : r \in \mathbb{R}\}$ is a family of operations independent over $M \cap \mathcal{O}^{(d)}$: for any $r \in \mathbb{R}$, we have $s_{A_r} \notin \langle s_{A_p} : p \in \mathbb{R} \setminus \{r\} \rangle_{\mathcal{O}^{(d)}}$. By Lemma 1.4 we are done. \square

Corollary 3.11. *There exists a clone C on \mathbb{N} such that for any $d \in \mathbb{N}$ there are 2^c clones D with $C \cap \mathcal{O}^{(d)} = D \cap \mathcal{O}^{(d)}$.*

Proof. Let C be the clone generated by all operations whose ranges are a subset of $\{0, 1\}$. To check, that this C satisfies the statement of the corollary, let $d \in \mathbb{N}$ and let C' be the clone generated by all at most d -ary operations whose ranges are contained in $\{0, 1\}$. Then $C \cap \mathcal{O}^{(d)} = C' \cap \mathcal{O}^{(d)}$ and C' is modest. Therefore, by Theorem 3.2 there exist 2^c many clones D with $D \cap \mathcal{O}^{(d)} = C' \cap \mathcal{O}^{(d)} = C \cap \mathcal{O}^{(d)}$. \square

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